UDC 532.526

ON A PARTICULAR BOUNDARY VALUE PROBLEM ARISING IN THE INVESTIGATION OF CLOSED STATIONARY SEPARATION ZONES IN AN INCOMPRESSIBLE FLUID

PMM Vol.39, No.5, 1975, pp.835-844

B. V. KOLOSOV and E. G. SHIFRIN
(Moscow)

(Received January 1, 1975)

The flow of a viscous incompressible fluid past a depression in the surface of a body at high Reynolds numbers was investigated in /1/ by the method of matching asymptotic expansions. An unusual, from the point of view of general theory, boundary value problem was formulated for boundary layer equations which in its main approximation defined the motion of fluid in the proximity of the separation zone boundary. Questions of uniqueness of the solution of this boundary value problem are considered below in relation to two approximations of equations of the boundary layer.

Let us consider the flow of a viscous incompressible fluid past a depression in the surface of a body, which induces a separation zone of finite dimensions (Fig. 1). We shall consider this zone – the circulation flow region – as developed (i.e. not infinitely splitting up and nonvanishing for $\mathrm{Re} \to \infty$). According to /1/ the over-all flow picture can be presented as follows. The oncoming stream (region *1* of translational motion) is separated from the region of circulation motion by the streamline ABC. The limit state of the flow of a viscous incompressible fluid in that region for $\mathrm{Re} \to \infty$ (if it exists) according to the Prandtl-Batchelor theorem /2/ is the flow of an inviscid fluid with constant vorticity ω (the viscosity coefficient is assumed constant). At reasonably high Re numbers at the boundary of constant vorticity region *3* there exist the following boundary layers: the mixing layer (region *2*) and the boundary layer next to the wall (region *4*). It was also assumed in /1/ that solutions of equations for inviscid flows can be used as the principal approximation for defining the flow which conforms to the complete Navier-Stokes equations /3/. These solutions are intended for matching boundary layers *2* and *4*.

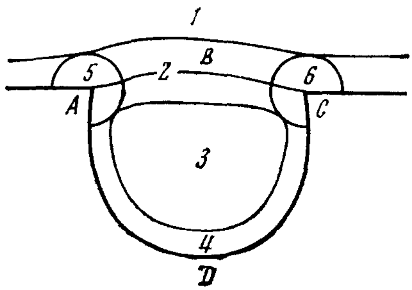

Fig. 1

We introduce in regions *2* and *4* (Fig. 1) boundary layer coordinates s and $N = n / \varepsilon$ (s is the length of arc of the contour $ABCDA$, measured from point A, n is the length of the instantaneous normal to $ABCDA$ external in relation to region *3*, and $\varepsilon = \mathrm{Re}^{-1/2}$). In these coordinates the boundary layer contained between the contour $ABCDA$ and the boundary of region *3* becomes a half-band $\sigma_1 = \{0 \leqslant s \leqslant s_A, N \leqslant 0\}$ (Fig. 2) which, owing to the flow cyclicity in the depression, can be periodically continued with respect to s over the whole negative half-plane, the part of the mixing layer lying above the dividing streamline ABC (Fig. 1) is transformed into the half-band $\sigma_2 = \{0 \leqslant s \leqslant s_C, N > 0\}$, and the depression wall into the half-band $\sigma_3 = \{s_C < s < s_A, N > 0\}$ (shaded in Fig, 2). Obviously $\sigma_2 \bigcup \sigma_3$ can be periodically continued with respect to s over the whole positive half-plane. (The plane with omitted half-bands (see Fig. 2) is called the σ-region.)

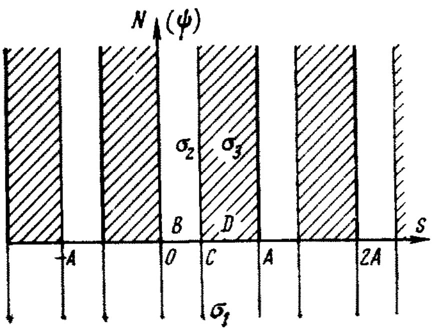

Fig. 2

The flow cyclicity manifests itself by the periodicity of boundary conditions. Along straight lines of the form $s = ks_A$ $(k = 0, \pm 1, \ldots)$ velocity distributions with respect to N are the same, and for $N \geqslant 0$ are determined by the profile of the oncoming boundary layer at point A. That profile may be assumed to be known. At segments of axis s $[ks_A + s_C \leqslant s < (k+1)s_A]$ the flow velocity vanishes (the condition of sticking).

Solution of this boundary value problem for the σ-region by the method of asymptotic expansion in accordance with /1/ must continuously merge with the solutions of problems of inviscid flow in regions 1 and 3 when $N \to +\infty$ and $N \to -\infty$, respectively.

The formulated boundary value problem (first considered in /1/) is not typical of the theory of the boundary layer owing to the shape of the region for which a solution is sought, and by the character of the boundary conditions, which makes it necessary to investigate the conditions of uniqueness of its solvability. Elucidation of the latter condition is needed in this case for the correct mathematical formulation of the boundary value problem. It will be shown below that generally the boundary condition for $N \to -\infty$ is overdetermining, and the solution (if it exists) of the problem in two approximations of the boundary layer theory makes it possible to determine all unknown parameters, viz. total pressure p_{03} and vorticity ω_3 in region 3, which is the asymptotics of solution for $N \to -\infty$.

That the considered boundary value problem (on condition of boundedness of its solution) does not actually necessitate the setting of a boundary condition for $N \to -\infty$, can be shown by the following example. Let us take advantage of the fact that the passing to variables s and ψ (ψ is the stream function) leaves the σ-region shape and the form of boundary conditions unchanged, while the boundary layer equations assume the form of equations of heat conduction (the Mises form /4/). Because of this we shall consider a boundary value problem, similar to the described above, for the equation of heat conduction.

It was found that in that case there exists a unique function $w(s, \psi) \in C_{s,\psi}^{1,2}(\sigma)$ bounded in σ and satisfying in it the following equation and boundary conditions

$$w_s = w_{\psi\psi} \tag{1}$$

$$w(ks_A, \psi) = \begin{cases} \alpha(\psi), & \psi \geqslant 0 \\ w[(k+1)s_A, \psi], & \psi < 0 \end{cases}, \quad w(s, 0) = 0, \quad ks_A + s_C \leqslant s < (k+1)s_A \tag{2}$$

where $(\alpha\ (\psi)$ is a known bounded function and $k = 0, \pm 1, \ldots$

To prove this we reduce the problem (1), (2) to an integral equation. For $\psi < 0$ we use the form of solution of the problem without initial conditions, i.e. the solution of the equation of heat conduction that satisfies boundary conditions specified for all $s > -\infty$ /5/

$$w(s, \psi) = (4\pi)^{-1/2} \int_{-\infty}^{s} \psi (s-\tau)^{-3/2} \exp\left[-\frac{\psi^2}{4(s-\tau)}\right] w(\tau, 0)\, d\tau \tag{3}$$

$$w(s, 0) = \begin{cases} \mu(s), & ks_A \leqslant s \leqslant ks_A + s_C \\ 0, & ks_A + s_C < s < (k+1)s_A \end{cases}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{4}$$

where formula (4) represent the boundary condition with, so far, unknown function $\mu(s)$. We denote $s_A \equiv A$ and $s_C \equiv C$, and set $s = 0$ in (3). Using the periodicity of the boundary condition (4), we write

$$w(0, \psi) = (4\pi)^{-1/2}\psi \sum_{k=1}^{\infty} \int_0^C \mu(\tau) F_k(\tau)\, d\tau, \qquad F_k(s) = (kA - s)^{-3/2} \times \exp\left[-\frac{\psi^2}{4(kA - s)}\right] \tag{5}$$

Let us assume that $\mu(s)$ is continuous along segment $[0, C]$ and that $\max|\mu(s)| = M$. Since the series

$$\sum_{k=1}^{\infty} |\mu(s)| F_k(s) \leqslant M \sum_{k=1}^{\infty} (kA - C)^{-3/2} \tag{6}$$

is uniformly convergent with respect to s (and also with respect to ψ), the change of the sequence of summation and integration in (5) yields for all $\psi < 0$

$$\varphi(\psi) \equiv w(0, \psi) = (4\pi)^{-1/2}\psi \int_0^C \mu(\tau) \sum_{k=1}^{\infty} F_k(\tau)\, d\tau \tag{7}$$

The solution of problem (1), (2) in the band $0 \leqslant s \leqslant C$ can be presented in the form of the Poisson's integral

$$w(s, \psi) = (4\pi s)^{-1/2}\left(\int_{-\infty}^{0} \varphi(\xi) \exp\left[-\frac{(\psi-\xi)^2}{4s}\right] d\xi + \int_0^{+\infty} \alpha(\xi) \exp\left[-\frac{(\psi-\xi)^2}{4s}\right] d\xi \right) \tag{8}$$

In virtue of the definition (7) of function $\varphi(\psi)$ and of estimate (6), it is permissible to alter in (8) the sequence of summation and integration with respect to ξ as well as that of integration with respect to ξ and τ. As the result, we obtain from (8) for $\psi = 0$ the integral equation

$$w(s, 0) \equiv \mu(s) = (4\pi)^{-1} \int_0^C \mu(\tau) K(s, \tau)\, d\tau + q(s) \tag{9}$$

$$K(s, \theta) \equiv s^{-1/2} \sum_{k=1}^{\infty} \int_{-\infty}^{0} \xi (kA - \theta)^{-3/2} \exp\left[-\frac{\xi^2}{4s}\left(1 + \frac{s}{kA - \theta}\right)\right] d\xi$$

$$q(s) \equiv (4\pi s)^{-1/2} \int_0^{+\infty} \alpha(\xi) \exp\left(-\frac{\xi^2}{4s}\right) d\xi$$

The definition of the kernel $K(s, \theta)$ clearly implies that

$$K(s,\theta) = (4s)^{1/2} \sum_{k=1}^{\infty} \frac{(kA-\theta)^{-1/2}}{kA+s-\theta} < M_1 \sqrt{s} \tag{10}$$

which shows that the integral operator in (9) is continuous with respect to s and, consequently, solutions (9) are continuous along segment $[0, C]$. This confirms the previously made assumption about the continuity of $\mu(s)$. It follows from (10) that the norm of $K(s, \psi)$ is bounded in $L_2(\sigma)$ and, consequently, (9) is a Fredholm equation of the second kind. According to the general theory the characteristic set of Fredholm operators is not greater than denumerable /6/ and in virtue of (9) depends parametrically on A and C. Hence the number $\lambda = 4\pi$ can be a characteristic not more than a denumerable number of times (depending on the values of A and C). This implies that solution (9) together with that of problem (1), (2) is unique for almost any finite A and C. This means that the conditions for $\psi \to +\infty$, function $\alpha(\psi)$, and the condition of solution periodicity uniquely define the solution in the negative half-plane, in particular for $\psi \to -\infty$.

Let us consider the boundary value problem in layer 2 and 4 in the two approximations of the boundary layer theory.

We represent the tangential and normal velocity components, denoted by u and v, respectively, and pressure p in the form of the following asymptotic expansions in $\varepsilon = \mathrm{Re}^{-1/2}$:

$$u(s, n; \varepsilon) = u_0(s, N) + \varepsilon u_1(s, N), \quad v(s, n; \varepsilon) = \varepsilon v_0(s, N) + \varepsilon^2 v_1(s, N), \quad p(s, n; \varepsilon) = p_0(s, N) + \varepsilon p_1(s, N)$$

The equations of motion can be represented to within terms of order ε in the following form /7/:

$$u_0 u_{0_s} + v_0 u_{0_N} + p_{0_s} - u_{0_{NN}} + \varepsilon(u_1 u_{0_s} + u_0 u_{1_s} - \varkappa N u_0 u_{0_s} + v_1 u_{0_N} + v_0 u_{1_N} + \varkappa u_0 v_0 - \varkappa N p_{0_s} + p_{1_s} - u_{1_{NN}} - \varkappa u_{0_N}) = 0 \tag{11}$$

$$p_{0_N} + \varepsilon(p_{1_N} - \varkappa u_0^2) = 0, \quad u_{0_s} + v_{0_N} + \varepsilon(u_{1_s} + v_{1_N} + \varkappa N v_{0_N} + \varkappa v_0) = 0$$

where $\varkappa(s)$ is the current curvature of the contour $ABCDA$ (streamlines of an inviscid flow).

Note. The zero equalities in (11) represent equalities to the sum of terms of expansion of the Navier-Stokes equations in powers of ε of order ε^2 and higher.

Taking this into consideration, we add to the last equation of system (11) the extraordinary terms $\varkappa\varepsilon^2 N v_{1_N}$ and $\varkappa\varepsilon^2 v_1$ and denote $(1 + \varkappa\varepsilon N)(v_0 + \varepsilon v_1) \equiv V$ and $u_0 + \varepsilon u_1 \equiv U$. This equation then assumes the form $U_s + V_N = 0$. We introduce the stream function $\psi_N = U$, and $-\psi_s = V$, and pass to the Mises variables s and ψ. As the result, the first two equations in (11) are transformed into equations of second order partial derivatives of the parabolic kind

$$(u_0^2/2)_s + \varepsilon u_{1_s} = u_0 (u_0^2/2)_{\psi\psi} + f_0(s) + \varepsilon[u_0 u_{1_{\psi\psi}} + b(u_{0_\psi}) u_{1_\psi} + d(u_0, u_{0_\psi}, u_{0_{\psi\psi}}) u_1 + g(s, \psi, u_{0_\psi})\varkappa + f_1(s)] \tag{12}$$

where $f_0(s)$ and $f_1(s)$ are continuous functions determined by the flow parameters at the boundaries of region 2 $(\psi \to \pm\infty)$. The boundary conditions

$$U(0, \psi) = U(A, \psi) = \alpha_0(\psi) + \varepsilon\alpha_1(\psi), \quad \psi \geqslant 0 \tag{13}$$

$$U(s, 0) = 0, \qquad C < s < A \tag{14}$$

are similar to (2).

The right-hand part of (13) defines the profile of the boundary layer velocities at point A of the wall AA'; for $\psi < +\infty$ function $\alpha_0(\psi)$ is bounded, while $\alpha_1(\psi)$ increases with increasing ψ. Functions $u_0(s, \psi)$ and $u_1(s, \psi)$ have the same properties when $\psi \to \pm\infty$, respectively.

When deriving the boundary condition for $\psi < 0$ it is necessary to consider the variation of the profile of velocities U in the ε-vicinity of angle points A and (regions 5 and 6 in Fig. 1). These variations are defined by the differences

$$U(C+\varepsilon, \psi) - U(C-\varepsilon, \psi) \equiv \beta_C(\psi) + \varepsilon\gamma_C(\psi),$$
$$U(A+\varepsilon, \psi) - U(A-\varepsilon, \psi) \equiv \beta_A(\psi) + \varepsilon\gamma_A(\psi)$$

(i.e. the difference between the values of U at exit from and entry to each region). According to /3/ $\beta_C(\psi) = \beta_A(\psi) \equiv 0$. Furthermore we consider in accordance with /1, 3/ that the complete asymptotics of the Navier-Stokes equations at a corner point is such that functions $\gamma_C(\psi)$ and $\gamma_A(\psi)$ are bounded and completely determined by the local problem of flow in regions 5 and 6 (Fig. 1) and, consequently, it is possible to set at the exit from these regions the problem of continuation of the boundary layer. Hence, if $U^0(s, \psi)$ denotes the part of solution of the considered problem which is continuous in $\sigma_1 \bigcup \sigma_2$ the third boundary condition is defined by

$$U(0, \psi) = U^0(A, \psi) + \gamma_C(\psi) + \gamma_A(\psi), \quad \psi < 0 \tag{15}$$

Let us elucidate the conditions of uniqueness of the solution of problem (12) – (15) in region $\sigma_1 \bigcup \sigma_2$.

Theorem. A unique solution of problem

$$L_{s,\psi}(w) \equiv a(s, \psi)w_{\psi\psi} + b(s, \psi)w_{\psi} + c(s, \psi)w - w_s = q(s, \psi) \tag{16}$$

$$w(0, \psi) = h(\psi), \quad \psi \geqslant 0; \quad w(0, \psi) = w(A, \psi) + \gamma_C(\psi) + \gamma_A(\psi), \ \psi < 0; \ w(s, 0) = 0, \ C < s < A \tag{17}$$

exists in region $\sigma_1 \bigcup \sigma_2$, if the following assumptions are satisfied:

1) coefficients a, b and c are bounded in $\sigma_1 \bigcup \sigma_2$, are continuous, and satisfy the Hölder conditions

$$|a(s, \psi') - a(s, \psi)| \leqslant M_1 |\psi' - \varphi|^{\lambda_1}, \quad |a(s', \psi) - a(s, \psi)| \leqslant M_1 |s' - s|^{\lambda_1}$$

$$|b(s, \psi') - b(s, \psi)| \leqslant M_1 |\psi' - \psi|^{\lambda_1}, \quad |c(s, \psi') - c(s, \psi)| \leqslant M_1 |\psi' - \psi|^{\lambda_1}, \quad \lambda_1 > 0$$

2) $a(s, \psi)$ satisfies the inequality $a(s, \psi) \geqslant \mu > 0$ for any $(s, \psi) \in \sigma_1 \bigcup \sigma_2$;

3) derivatives a_ψ, $a_{\psi\psi}$ and b_ψ exist in $\sigma_1 \bigcup \sigma_2$, are bounded, continuous, and satisfy the Hölder condition with respect to ψ;

4) function $q(s, \psi)$ is continuous in $\sigma_1 \bigcup \sigma_2$ and for $0 \leqslant s \leqslant A$ satisfies the estimate

$$|q(s, \psi)| \leqslant M_2 |\psi|^{\lambda_2}, \quad \lambda_2 > 0 \tag{18}$$

5) function $h(\psi)$ is continuous for $0 \leqslant \psi \leqslant +\infty$ and, also, satisfies estimate (18); $\gamma_C(\psi)$ and $\gamma_A(\psi)$ are bounded and continuous.

Proof. Conditions (1) and (2) of the theorem ensure the existence of a unique fundamental solution $Z(s,\psi;\tau,\xi)$ of the homogeneous equation (16) in the layer $\overline{H}=\{0\leqslant s\leqslant C, -\infty\leqslant\psi\leqslant+\infty\}$ /8/.

Let us prove that

$$w(s,\psi)=-\int_0^s d\tau\int_{-\infty}^{+\infty} Z(s,\psi;\tau,\xi)\,q(\tau,\xi)\,d\xi+\int_{-\infty}^{+\infty} Z(s,\psi;\tau,\xi)\times \Phi(\xi)\,d\xi\equiv V_1(s,\psi)+V_2(s,\psi) \tag{19}$$

is the solution of Eq. (16) in layer $\overline{H}$, which satisfies the initial condition

$$w(0,\psi)=\Phi(\psi)\equiv\begin{cases}h(\psi), & \psi\geqslant 0\\ \varphi(\psi), & \psi<0\end{cases} \tag{20}$$

where $\varphi(\psi)$, a function yet to be determined, is continuous and satisfies (18).

On the basis of estimates

$$|Z(s,\psi;\tau,\xi)|<M_{3}{}^{*}{}_{,-1/2},\quad \left|\frac{\partial Z(s,\psi;\tau,\xi)}{\partial\psi}\right|<M^{*}{}_{3,-1}, \tag{21}$$

$$\left|\frac{\partial^2 Z(s,\psi;\tau,\xi)}{\partial\psi^2}\right|<M^{*}{}_{3,-3/2},$$

$$\left|\frac{\partial Z(s,\psi;\tau,\xi)}{\partial s}\right|<M_3{}^{*}{}_{,-3/2},\quad M_{3,\lambda}\equiv M_3(s-\tau)^{\lambda}\exp\left[-\frac{\mu_1(\psi-\xi)^2}{s-\tau}\right]$$

obtained in /8/ and of condition (4) of this theorem we have

$$|J(s,\psi;\tau)|\equiv\left|\int_{-\infty}^{+\infty} Z(s,\psi;\tau,\xi)\,q(\tau,\xi)\,d\xi\right|<M_4,\quad |V_1(s,\psi)|<M_4 s$$

which implies that $V_1(s,\psi)$ is continuous in $\overline{H}$ and satisfies the initial zero condition. Furthermore, the integrals

$$\frac{\partial J}{\partial\psi}=\int_{-\infty}^{+\infty}\frac{\partial Z}{\partial\psi}q(\tau,\xi)\,d\xi,\quad \frac{\partial^2 J}{\partial\psi^2}=\int_{-\infty}^{+\infty}\frac{\partial^2 Z}{\partial\psi^2}q(\tau,\xi)\,d\xi,$$

$$\frac{\partial J}{\partial s}=\int_{-\infty}^{+\infty}\frac{\partial Z}{\partial s}q(\tau,\xi)\,d\xi$$

in virtue of (21) are continuous with respect to s and ψ, and are uniformly convergent.

From this, using the following property of function $Z(s,\psi;\tau,\xi)$:

$$\lim_{s\to\tau+0}\int_{-\infty}^{+\infty} Z(s,\psi;\tau,\xi)\,q(\tau,\xi)\,d\xi=q(s,\psi)$$

we find that $V_1(s,\psi)$ satisfies the equation $L_{s,\psi}(V_1(s,\psi))=q(s,\psi)$. The proof that $V_2(s,\psi)$ satisfies the equation $L_{s,\varphi}(V_2(s,\psi))=0$ and the initial condition (20) is similar.

For $s=C-0$ from (19) we obtain

$$w(C,\psi)=-\int_0^C d\tau\int_{-\infty}^{+\infty} Z(c,\psi;\tau,\xi)\,q(\tau,\xi)\,d\xi+\int_{-\infty}^{+\infty} Z(C,\psi;0,\xi)\,\Phi(\xi)\,d\xi \tag{22}$$

Condition (3) of the theorem ensures the existence of operator $L^*_{\tau,\xi}$,

$$L^*_{\tau,\xi}(v) \equiv \frac{\partial^2 (a(\tau,\xi)v)}{\partial\xi^2} - \frac{\partial (b(\tau,\xi)v)}{\partial\xi} + c(\tau,\xi)v - \frac{\partial v}{\partial\tau}$$

conjugate of $L_{s,\psi}$, as well as the presentation of any solution $w(s,\psi)$ of the homogeneous equation (16) in the half-band $\bar{Q} = \{C \leqslant s < A, -\infty \leqslant \psi < 0\}$ in the form /8/

$$w(s,\psi) = \int_{-\infty}^{0} w(C,\xi)G(s,\psi;C,\xi)\,d\xi + \int_{C}^{A} w(\tau,0)\times$$
$$\frac{\partial[a(\tau,0)G(s,\psi;\tau,0)]}{\partial\xi}\,d\tau$$
$$G(s,\psi;\tau,\xi) \equiv Z(s,\psi;\tau,\xi) - v(s,\psi;\tau,\xi)$$

Function $v(s,\psi;\tau,\xi)$ is subordinated to conditions:

1°. $v(s,\psi;\tau,\xi)$ is determinate in $\bar{Q}$ and for $\tau < s$ satisfies the equation $L_\tau{}^*{}_{,\xi}(v) = 0$;

2°. $\lim\limits_{s\to\tau} v(s,\psi;\tau,\xi) = 0$;

3°. $v(s,0;\tau,\xi) = Z(s,0;\tau,\xi)$

We seek function $v(s,\psi;\tau,\xi)$ in the form of the potential of the double layer of unknown density $\omega(s;\tau,\xi)$ /8/

$$v(s,\psi;\tau,\xi) = \int_{\tau}^{s} d\theta \int_{-\infty}^{0} P(Z(s,\psi;\theta,\zeta))\,\omega(\theta;\tau,\xi)\,d\zeta \tag{23}$$

$$P(Z(s,\psi;\tau,\xi)) \equiv \frac{\partial[a(\tau,\xi)Z(s,\psi;\tau,\xi)]}{\partial\xi} - b(\tau,\xi)Z(s,\psi;\tau,\xi) \tag{24}$$

Equation (24) implies the estimate /8/

$$|P(Z(s,\psi;\tau,\xi))| < M_5(s-\tau)^{-1+\lambda}\exp\left[-\frac{\mu_2(\psi-\xi)^2)}{s-\tau}\right] \tag{25}$$

where (M_5, λ and μ_2 are positive constants.

According to /9/ the relationship

$$\lim_{\psi\to 0} v(s,\psi;\tau,\xi) = {}^1\!/_2\,\omega(s;\tau,\xi) + v(s,0;\tau,\xi)$$

(theorem about the potential jump of a double layer) is valid. Hence, satisfying condition 3°, we obtain for the determination of $\omega(s;\tau,\xi)$ the integral Volterra equation of the second kind

$$\frac{1}{2}\omega(s;\tau,\xi) + \int_{\tau}^{s} d\theta \int_{-\infty}^{0} P(Z(s,0;\theta,\zeta))\,\omega(\theta;\tau,\xi)\,d\zeta = Z(s,0;\tau,\xi)$$

We seek its solution in the form of series

$$\omega(s;\tau,\xi) = \sum_{m=1}^{\infty} \omega_m(s;\tau,\xi) \tag{26}$$

$$\omega_1 = Z(s,0;\tau,\xi); \quad \omega_{m+1} = \int_{\tau}^{s} d\theta \int_{-\infty}^{0} P(Z(s,0;\theta,\zeta))\,\omega_m(\theta;\tau,\xi)\,d\zeta,$$
$$m = 1, 2, \ldots$$

Let us prove the convergence of series (26) for $s > \tau$. In virtue of (21) and (25) we have

$$|\omega_1(s;\tau,\xi)| < M_3 (s-\tau)^{-1/2} \exp\left[-\frac{\mu_1\xi^2}{s-\tau}\right]$$

$$|\omega_2(s;\tau,\xi)| < M_6^2 \int_\tau^s (s-\theta)^{-1/2+\lambda} d\theta \int_{-\infty}^{+\infty} [(s-\theta)(\theta-\tau)]^{-1/2} \times$$
$$\exp\left[-\mu_3\left(\frac{\zeta^2}{s-\theta}+\frac{\xi^2}{\theta-\tau}\right)\right]d\zeta = M_6^2 \sqrt{\frac{\pi}{\mu_3}} \frac{(s-\tau)^\lambda}{(1/2+\lambda)} \times$$
$$\exp\left[-\frac{\mu_3\xi^2}{s-\tau}\right]$$

$$|\omega_3(s;\tau,\xi)| < \frac{M_6^2\sqrt{\pi/\mu_3}}{(1/2+\lambda)} \int_\tau^s (s-\theta)^{\lambda-1/2}(\theta-\tau)^{\lambda+1/2} d\theta \times$$
$$\int_{-\infty}^{+\infty} [(s-\theta)(\theta-\tau)]^{-1/2} \exp\left[-\mu_3\left(\frac{\zeta^2}{s-\theta}+\frac{\xi^2}{\theta-\tau}\right)\right]d\zeta =$$
$$M_6^3 \frac{\pi(s-\tau)^{2\lambda+1/2}}{\mu_3(1/2+\lambda)} \exp\left[-\frac{\mu_3\xi^2}{s-\tau}\right] \int_0^1 \eta^{\lambda+1/2}(1-\eta)^{\lambda-1/2} d\eta =$$
$$M_6^3 \frac{\pi}{\mu_3}\cdot \frac{\Gamma^2(\lambda+1/2)(s-\tau)^{2\lambda+1/2}}{\Gamma[2(\lambda+1/2)](2\lambda+1)} \exp\left[-\frac{\mu_3\xi^2}{s-\tau}\right]$$

where $\Gamma(\alpha)$ is the gamma function.

By induction with respect to m it is possible to show that for $m \geqslant 3$

$$|\omega_m(s;\tau,\xi)| < M_6^m \left(\frac{\pi}{\mu_3}\right)^{\frac{m-1}{2}} \frac{\Gamma^{m-1}(\lambda+1/2)(s-\tau)^l}{\Gamma[(m-1)(\lambda+1/2)](m-1)(\lambda+1/2)} \times \tag{27}$$
$$\exp\left[-\frac{\mu_3\xi^2}{s-\tau}\right]$$
$$l = \frac{(m-1)(2\lambda+1)-1}{2}$$

Since for $m(\lambda+1/2) > 2$ we have $\Gamma[m(\lambda+1/2)] \geqslant [m(\lambda+1/2)]!$, hence (27) implies the absolute and uniform convergence of series (26) for $s > \tau$, as well as the estimate

$$|\omega(s;\tau,\xi)| < M_7 (s-\tau)^{-1/2} \exp\left[-\frac{\mu_3\xi^2}{s-\tau}\right] \tag{28}$$

Taking into account (25), from (23) and (28) we obtain

$$|v(s,\psi;\tau,\xi)| < M_8 (s-\tau)^\lambda \exp\left[-\frac{\mu_3(\psi-\xi)^2}{s-\tau}\right]$$

hence $v(s,\psi;\tau,\xi)$ satisfies condition 2°. Since by condition (3) of the theorem $L^*_{\tau,\xi}(Z(s,\psi;\tau,\xi)) = 0$ (see /8/), the fulfilment of condition 1° is evident. Thus the existence of a unique Green's function $G(s,\psi;\tau,\xi)$ is established.

Let us consider function $\varphi_C(\psi) \equiv w(C-0,\psi)$ for $-\infty \leqslant \psi < 0$, where $w(C-0,\psi)$ is determined by (22). A proof similar to that of (19) shows that

$$w(s,\psi) = -\int_C^s d\tau \int_{-\infty}^0 G(s,\psi;\tau,\xi)\, q(\tau,\xi)\, d\xi + \int_{-\infty}^0 G(s,\psi;C,\xi)\,[\varphi_C(\xi)+\gamma_C(\xi)]\, d\xi$$

is the solution of Eq. (16) in the half-band $\bar{Q}$, which satisfies boundary conditions $w(C, \psi) = \varphi_C(\psi)$ for $-\infty \leqslant \psi < 0$ and $w(s, 0) = 0$. Allowing for the discontinuity of the solution of problem (16), (17) for $s = A + 0$ and $\psi < 0$, for the determination of $\varphi(\psi)$ we obtain

$$\varphi(\psi) = \int_{-\infty}^{0} K(\psi, \zeta; A, C)\varphi(\zeta)\,d\zeta + r(\psi; A, C) + \gamma_A(\psi) \tag{29}$$

where

$$K(\psi, \zeta; A, C) \equiv Z(C, \psi; 0, \zeta)\int_{-\infty}^{0} G(A, \psi; C, \xi)\,d\xi \tag{30}$$

$$r(\psi; A, C) \equiv -\int_0^C d\tau \int_{-\infty}^{0} G(A, \psi; C, \xi)\left(\int_{-\infty}^{+\infty} Z(C, \psi; \tau, \zeta)\,q(\tau, \zeta)\,d\zeta\right)d\xi - \tag{31}$$

$$\int_C^A d\tau \int_{-\infty}^{0} G(A, \psi; \tau, \xi)\,q(\tau, \xi)\,d\xi + \int_{-\infty}^{0} G(A, \psi; C, \xi)\times$$

$$\left(\int_0^{+\infty} Z(C, \psi; 0, \zeta)\,h(\zeta)\,d\zeta\right)d\xi + \int_{-\infty}^{0} K(\psi, \zeta; A, C)\,\gamma_C(\zeta)\,d\zeta$$

(the change of the order of integration in (29) and in the first and second terms of (31) is justified by the second of estimates (21). It follows from (21) and (30) that the norm of kernel $K(\psi, \zeta; A, C)$ is bounded in $L_2(\sigma_1 \cup \sigma_2)$, hence (29) is a Fredholm equation of the second kind. The resolvent of its kernel is a meromorphic function whose poles are roots of the polynomial /6/

$$D(\nu) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_n \nu^n, \quad c_0 = 1, \quad c_n = \int_0^{+\infty} B_{n-1}(\psi, \psi)\,d\psi, \quad n > 0$$

$$B_n(\psi, \zeta; A, C) = c_n K(\psi, \zeta; A, C) - n\int_0^{+\infty} K(\psi, \theta; A, C)\,B_{n-1}(\theta, \zeta; A, C)\,d\theta$$

It is clear that the coefficients (and also the roots) of the polynomial $D(\nu)$ depend on A and C as parameters. Hence the number $\nu = 1$ can be a characteristic one not more than in a denumerable set of finite values of A and C, and, consequently, solutions of Eq. (29) and of problem (16), (17) are unique for almost any A and C.

Remarks. 1°. Since coefficients in (12), which depend on the first approximation solution $u_0(s, \psi)$ and its derivatives, are bounded continuous functions, hence they satisfy conditions (1) and (3) of the above theorems. Functions $g(s, \psi, u_{0\psi})$ in (12), and $\alpha_0(\psi)$, and $\alpha_1(\psi)$ in (13) satisfy conditions (4) and (5), respectively, as well as estimate (18) with $\lambda_2 = 1$. In virtue of (14) the inequality in condition (2) is not satisfied for Eq. (12). However, if the remark related to the derivation of system (11) is taken into account, it is possible to substitute

$$U(s, 0) = O(\varepsilon^p) > 0 \quad (p \geqslant 2), \quad C < s < A \tag{32}$$

for (14), and consider that condition (2) of the theorem is satisfied also for (12).

2°. According to /10/ it is possible to reduce the analysis of uniqueness of the boundary problem solution of a quasi-linear equation to the analogous analysis of the solution of some linear equation with zero boundary conditions. Therefore the uniqueness of solution follows from the above theorem, if the existence of a first approximation solution of the problem defined by (12), (13), (15) and (32) is assumed.

3°. The theorem proved above with allowance for (32) implies the existence and uniqueness of second approximation solution of problem (12) – (15).

REFERENCES

1. Neiland, V. Ia. and SYCHEV, V. V., On the theory of flow in stationary separation zones. Uch. Zap. TsAGI, Vol. 1, No. 1, 1970.
2. Batchelor, G. K., On steady laminar flow with closed streamlines at large Reynolds number. J. Fluid Mech., Vol. 1, pt. 2, 1956.
3. Matveeva, N. S. and Neiland, V. Ia., The laminar boundary layer close to an angle point of a body. Izv. Akad. Nauk SSSR, MZhG, No. 4, 1967.
4. Kochin, N. E., Kibel', I. A. and Roze, N. V., Theoretical Hydromechanics, Vol. 2, Fizmatgiz, Moscow, 1963.
5. Tikhonov, A. N. and Samarskii, A. A., Equations of Mathematical Physics. (English translation), Pergamon Press, Book No. 10226, 1963.
6. Integral Equations. Mathematical Reference Library. "Nauka", Moscow, 1968.
7. Van Dyke, M., Theory of compressible boundary layer in the second approximation. Collection: Investigation of Hypersonic Flows (Russian translation), "Mir", Moscow, 1964.
8. Il'in, A. M., Kalashnikov, A. S. and Oleinik, O. A., Second order linear equations of the parabolic kind. Uspekhi Matem. Nauk, Vol. 17, No. 3 (105), 1962.
9. Slobodetskii, L. N., Potential theory for parabolic equations. Dokl. Akad. Nauk SSSR, Vol. 103, No. 1, 1955.
10. Courant, R., Differential Equations with Partial Derivatives. "Nauka", Moscow, 1964.

Translated by J. J. D.

UDC 532.529

ON THE EQUATIONS OF MOTION OF A LIQUID WITH BUBBLES

PMM Vol. 39, No. 5, 1975, pp. 845-856

O. V. VOINOV and A. G. PETROV
(Moscow)

(Received June 18, 1974)

An arbitrary irrotational flow of perfect incompressible liquid containing a considerable number of spherical gas bubbles is considered. Two methods of averaging exact characteristics of the motion of bubbles in the liquid, viz. by volume and by bubble centers, are introduced. Formulas relating the average quantities of two different kinds are derived. The boundary value problem for the mean potential is formulated on the basis of the exact boundary value problem for the velocity potential. The obtained equation for the potential in the particular case of unbounded liquid with low concentration of bubbles coincides with that derived in /1/.

It is shown that dynamic equations for the average characteristics of moving bubbles accurate to within the product of volume concentration by the velocity